A Branching Process Approach to Power Markets

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Spikes in electricity prices

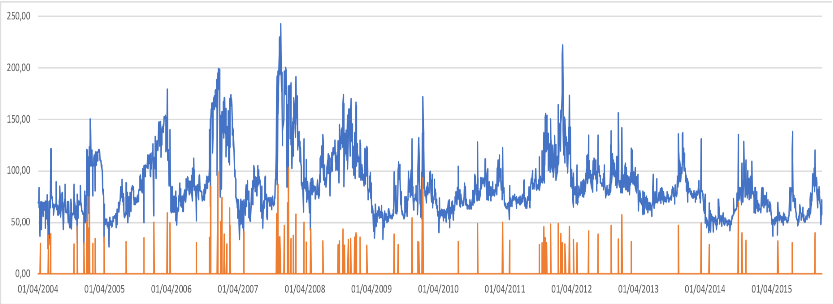

FIGURE – Single National Price (PUN at 7PM only working days) of electricity in Italy between April 2004 and December 2015 and large positive fluctuations

We observe 96 positive jumps over 140 months, that is one jump each 30 working days. However, the distribution is far to be homogeneous.

Poisson versus self-exciting structure

Rejection of Poisson framework

We first test the goodness of fit of a pure Poisson distribution using the Kolmogorov Smirnov statistics. The value of the test is 1.821 that is larger than the critical value 1.628 for a significance level 1%.

Poisson versus self-exciting structure

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Lemma

Let N be a non-homogeneous Poisson process with rate μ. Denote by T_i the increasing sequence of arrival times of the jumps of N. Then, fix $t > 0$ conditional on N_t, the vector $(T_1, T_2, \ldots, T_{N_t})$ has the same law as $(U_{(1)}, U_{(2)}, \ldots, U_{(N_t)})$, i.e. the order statistics built from uniform IID random variables with density $\frac{\mu(s)\mathbb{I}_{s\in[0,t]}}{\int_0^t \mu(u)du}$.

Acceptance of self-exciting framework

We test the goodness of fit of a pure self-exciting jumps framework, that is the intensity μ is proportional to the price of electricity itself. The value of the Kolmogorov-Smirnov test is $1,06$ that is lower than the critical value 1.224 for a significance level 10%.

Kolmogorov-Smirnov plot

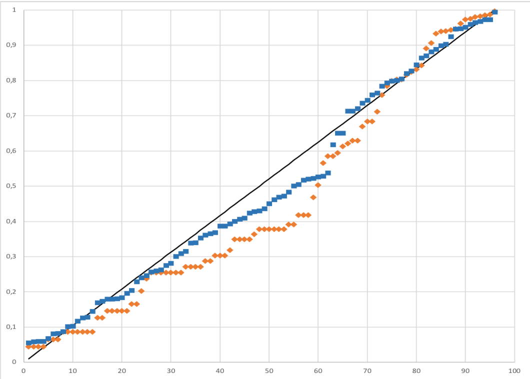

FIGURE – QQ-plot of jumps arrival. Pure Poisson case in orange. Self-exciting case in blue.

Power Price Modeling

We will assume then the spot price process $S(t)$ to evolve according to the basic dynamics :

$$S(t) = B(t) + Y(t),$$

where $B(t)$ is a seasonality function of deterministic type and the process $Y(t)$ is a superposition of the factors $X_i(t)$:

$$Y(t) = \sum X_i(t),$$

The main objective is to propose new candidates for the evolution of the factors X including self-exciting structure. We propose to look at the class of continuous state branching processes with immigration.

Branching property

Branching property :

A process X has the Branching Property if for any t and x, y in the state space of X, X_t^{x+y} is equal in law to the independent sum of X_t^x and X_t^y.

If a process X can be decomposed as $X = X^{(1)} + X^{(2)}$ where for $i = 1, 2$, $X^{(i)}$ satisfying the same SDE with $X_0 = X_0^{(1)} + X_0^{(2)}$, then the process is said a **branching process**.

We have the following result, see Kawazu and Watanabe (1971).

Generator

Markov process X with state space $\mathbb{R}_+$ with Branching mechanism :

$$\Psi(q) = \beta\, q + \frac{1}{2}\sigma^2\, q^2 + \int_0^\infty (e^{-q\,\zeta} - 1 + q\,\zeta)\,\pi(d\zeta),$$

with $\sigma \geq 0$, $\beta \in \mathbb{R}$ and π being a Lévy measure such that $\int_0^\infty (\zeta \wedge \zeta^2)\,\pi(d\zeta) < \infty$.
The CBI process X has as generator the operator $\mathcal{L}$ acting on $C_0^2(\mathbb{R}_+)$ as

$$\mathcal{L}f(x) = \frac{\sigma^2}{2}\, x f''(x) - \beta\, x f'(x) + x \int_0^\infty \big(f(x+\zeta) - f(x) - \zeta f'(x)\big)\,\pi(d\zeta).$$

Dawson Li (2006) representation

Integral representation

The previous generator admits the following semigroup (Hille-Yosida theorem).

$$X_t = -\int_0^t \int_0^{X_s} a\,du\,ds + \sigma \int_0^t \int_0^{X_s} W(ds,du) + \int_0^t \int_0^{X_{s-}} \int_{\mathbb{R}+} \zeta\,\widetilde{N}(ds,du,d\zeta),$$

- $W(ds,du)$: white noise on $\mathbb{R}^2_+$ with intensity $ds\,dv$,
- $\widetilde{N}(ds,du,d\zeta)$: compensated Poisson random measure on $\mathbb{R}^3_+$ with intensity $ds\,du\,\pi(d\zeta)$, Besides, W and N are independent of each other.

Main problem : the process converges to 0 if $a > 0$ or to ∞ otherwise. As a consequence it is not ergodic.

Continuous state branching process with immigration

Integral representation

$$
\begin{aligned}
X_t &= \int_0^t a(b - X_s)\, ds + \sigma \int_0^t \int_0^{X_s} W(ds, du) \\
&+ \gamma \int_0^t \int_0^{X_{s-}} \int_{\mathbb{R}+} \zeta\, \widetilde{N}(ds, du, d\zeta) + \overline{\gamma} \int_0^t \int_{\mathbb{R}+} \zeta\, M(ds, d\zeta),
\end{aligned}
$$

- $M(ds, d\zeta)$: compensated Poisson random measure on $\mathbb{R}_+^2$ with intensity $ds\overline{\pi}(d\zeta)$,

The process will be exponential ergodic if $a > 0$.

$$
\begin{aligned}
\mathcal{L}f(x) &= a(b-x)f'(x)\frac{\sigma^2}{2}xf''(x) + x\int_0^\infty \big(f(x+\zeta) - f(x) - \zeta f'(x)\big)\,\pi(d\zeta) \\
&+ \int_0^\infty \big(f(x+\zeta) - f(x)\big)\,\overline{\pi}(d\zeta)
\end{aligned}
$$

Continuous state branching process with immigration (CBI)

CBI (Kawazu & Watanabe 1971) of branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$: Markov process X with state space $\mathbb{R}_+$ verifying

$$\mathbb{E}^x\left[e^{-p X_t}\right] = \exp\left[-x\, v(t,p) - \int_0^t \Phi\big(v(s,p)\big)\, ds\right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

, and Ψ and Φ are functions on $\mathbb{R}_+$ given by

$$\Psi(q) = a\, q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-q\,\zeta} - 1 + q\,\zeta)\, \pi(d\zeta),$$

$$\Phi(q) = ab\, q + \int_0^\infty (1 - e^{-q\,\zeta})\, \overline{\pi}(d\zeta),$$

with $\sigma, ab \geq 0$, $a \in \mathbb{R}$ and $\pi, \overline{\pi}$ being two Lévy measures such that $\int_0^\infty (\zeta \wedge \zeta^2)\, \pi(d\zeta) < \infty$ and $\int_0^\infty (1 \wedge \zeta)\, \overline{\pi}(d\zeta) < \infty$.

Link to Hawkes process

- When $\sigma = \overline{\gamma} = 0$ and $\pi(d\zeta) = \delta_1(d\zeta)$, then X is given by

$$X_t = X_0 - \int_0^t \big(a + \pi(\mathbb{R}^+)\big) X_s \, ds + \int_0^t \int_0^{X_{s-}} N(ds, du) \tag{1}$$

which is the intensity of Hawkes process

$$\int_0^t \int_0^{X_{s-}} N(ds, du),$$

N being the Poisson random measure with intensity $ds\, du$.

The α-CIR model setup : Integral representation (Dawson-Li)

Integral form by using the random fields

$$X_t = X_0 + \int_0^t a\,(b - X_s)\,ds + \sigma \int_0^t \int_0^{X_s} W(ds, du) + \sigma_Z \int_0^t \int_0^{X_{s-}} \int_{\mathbb{R}+} \zeta\,\widetilde{N}(ds, du, d\zeta), \tag{2}$$

- $W(ds, du)$: white noise on $\mathbb{R}_+^2$ with intensity $dsdu$,
- $\widetilde{N}(ds, du, d\zeta)$: compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $ds\,du\,\mu(d\zeta)$,
- $\mu(d\zeta)$: a Lévy measure satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\,\mu(d\zeta) < \infty$.

We choose the Lévy measure to be

$$\mu(d\zeta) = -\frac{1_{\{\zeta>0\}}\,d\zeta}{\cos(\pi\alpha/2)\,\Gamma(-\alpha)\,\zeta^{1+\alpha}}, \quad 1 < \alpha < 2,$$

For existence and uniqueness of the solution see Dawson and Li (2012), Theorem 3.1 and Li and Ma (2015) Theorem 2.1.

The α-CIR model setup

We consider the following *usual* SDE

$$X_t = X_0 + \int_0^t a\,(b - X_s)\,ds + \sigma \int_0^t \sqrt{X_s}\,dB_s + \sigma_Z \int_0^t X_{s-}^{1/\alpha}\,dZ_s \tag{3}$$

- $B = (B_t, t \geq 0)$ a Brownian motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive α-stable compensate Lévy process with parameter $\alpha \in (1, 2]$ with
$$\mathbb{E}\left[e^{-q\,Z_t}\right] = \exp\left\{-\frac{t\,q^{\alpha}}{\cos(\pi\,\alpha/2)}\right\}, \quad q \geq 0.$$
- B and Z are independent.

Z_t follows the α-stable distribution $S_\alpha(t^{1/\alpha}, 1, 0)$ with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift.
The existence of a unique strong solution for the SDE (3) follows from Fu and Li (Theorem 5.3, 2010).

Simulation of processes Z and X with different α

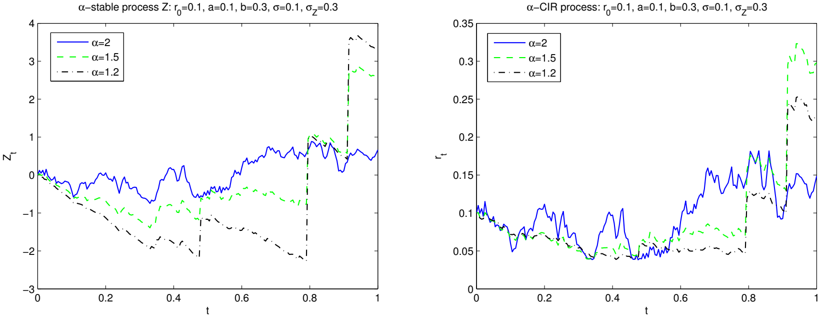

FIGURE – Three parameters of α : 2 (blue), 1.5 (green) and 1.2 (black).

Similar properties with CIR model I

Boundary condition :

The point 0 is an inaccessible boundary if and only if $2\,a\,b \geq \sigma^2$. In particular, a pure jump α-CIR process with $ab > 0$ never reaches 0 since $\sigma = 0$.

Ergodic law :

The process is exponentially ergodic, the limit distribution denoted by r_∞ satisfies

$$\mathbb{E}\left[e^{-pX_\infty}\right] = \exp\left\{-\int_0^p \frac{abq}{aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha} dq\right\}.$$

Similar properties with CIR model II

Branching property :

r can be decomposed as $X = X^{(1)} + X^{(2)}$ where for $i = 1, 2$, $X^{(i)}$ is an α-CIR$(a, b^{(i)}, \sigma, \sigma_Z, \alpha)$ process such that $X_0 = X_0^{(1)} + X_0^{(2)}$ and $b = b^{(1)} + b^{(2)}$.

See Dawson and Li (2006).
This property is a direct consequence of

- linearity of integrals,
- homogeneity of measures.

Equivalence of two representations

Then the root representation (3) and the integral representation (2) are equivalent in the following sense :

- The solutions of the two equations have the same probability law.
- On an extended probability space, they are equal almost surely.

See Theorem 9.32 in Li (2011).
The equivalence is useful since we have two ways to study the properties of the model.

Power Price Modeling

We will assume then the spot price process $S(t)$ to evolve according to the basic dynamics :

$$S(t) = B(t) + \sum X_i(t),$$

$$\begin{aligned} X_i(t) &= X_i(0) + \int_0^t a\,(b_i - X_i(s))\,ds + \sigma_i \int_0^t \int_0^{X_i(s)} W(ds,du) \\ &\quad + \gamma_i \int_0^t \int_0^{X_i(s-)} \int_{\mathbb{R}+} \zeta\,\widetilde{N}_i(ds,du,d\zeta). \end{aligned}$$

This kind of dynamics extends that proposed by F.E. Benth, J. Kallsen and T. Meyer-Brandis, by keeping the basic features of an Ornstein-Uhlenbeck process driven by a subordinator, but it introduces the self exciting properties in a direct and natural way.

Finite activity case

$$\begin{aligned} X_i(t) &= X_i(0) + \int_0^t [ab_i - \tilde{a}_i X_i(s)]\, ds + \sigma_i \int_0^t \int_0^{X_i(s)} W(ds, du) \\ &+ \gamma_i \int_0^t \int_0^{X_i(s-)} \int_{\mathbb{R}+} \zeta\, N_i(ds, du, d\zeta), \end{aligned}$$

for $t \geq 0$, where :

$$\tilde{a}_i(t) = a + \sigma_i \int_{\mathbb{R}+} \zeta\, \pi_i(d\zeta).$$

The last expression shows how a different mean-reversion speed for each factor Y_i can arise even if a common mean-reversion speed is assigned.

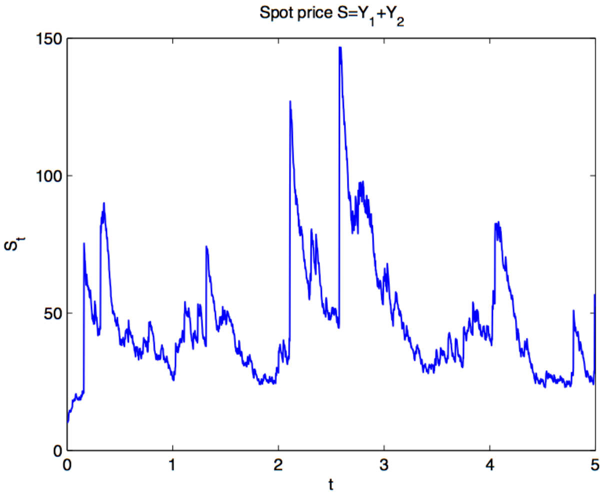

FIGURE – The Power Spot Price Dynamics.

Locally equivalent CIR process with jumps

- Consider the α-CIR process with initial value X_0 and introduce

$$P_t = X_0 + \int_0^t a\,(b - P_s)\,ds + \sigma \int_0^t \int_0^{P_s} W(ds, du) + \gamma \int_0^t \int_0^{X_0} \int_{\mathbb{R}+} \zeta\, \widetilde{N}(ds, du, d\zeta),$$

  where the processes W and $\widetilde{N}$ are (almost) the same as in (3).
- the above CIR process with jumps can be written as

$$dP_t = X_0 + a\,(b - P_t)\,dt + \sigma\sqrt{P_t}\,dB_t + \sigma_Z\ \sqrt[\alpha]{X_0}\,dZ_t,$$

- The implicit negative drifts lead to a linear decay for P_t while an exponential decay for X_t : when γ increases, the decreasing drift plays a more important role in α-CIR than in equivalent CIR with jumps.

Comparison between α-CIR and CIR with α-stable jumps (continued)

- Separating small and large jumps in CIR with jumps, we get

$$P_t = X_0 + \int_0^t a\left(b - \frac{\gamma X_0 \,\Theta(\alpha,y)}{a} - P_s\right) ds + \sigma \int_0^t \int_0^{P_s} W(ds,du) + \gamma \int_0^t \int_0^{X_0} \int_0^y \zeta\, \widetilde{N}(ds,du,d\zeta) + \gamma \int_0^t \int_0^{X_0} \int_y^\infty \zeta\, N(ds,du,d\zeta),$$

where

$$\Theta(\alpha,y) = \frac{2}{\pi}\,\alpha\,\Gamma(\alpha-1)\,\frac{\sin(\pi\alpha/2)}{y^{\alpha-1}}.$$

- In a similar way, the α-CIR process can be written as

$$X_t = X_0 + \int_0^t \widetilde{a}(\alpha,y)\left(\widetilde{b}(\alpha,y) - X_s\right) ds + \sigma \int_0^t \int_0^{X_s} W(ds,du) + \gamma \int_0^t \int_0^{X_{s-}} \int_0^y \zeta\, \widetilde{N}(ds,du,d\zeta) + \gamma \int_0^t \int_0^{X_{s-}} \int_y^\infty \zeta\, N(ds,du,d\zeta),$$

where

$$\widetilde{a}(\alpha,y) = a + \gamma\,\Theta(\alpha,y), \quad \widetilde{b}(\alpha,y) = \frac{ab}{a+\gamma\,\Theta(\alpha,y)}.$$

Change of Probability

Proposition :

Let X be a $CBI(a, b, \sigma, \gamma, \pi)$ process under the probability measure $\mathbb{P}$ and assume that the filtration $\mathbb{F}$ is generated by the random fields W and $\widetilde{N}$. Fix $\eta \in \mathbb{R}$ and $\theta \in \mathbb{R}_+$, and define

$$U_t := \eta \int_0^t \int_0^{X_s} W(ds, du) + \int_0^t \int_0^{X_{s-}} \int_0^\infty (e^{-\theta\,\zeta} - 1)\,\widetilde{N}(ds, du, d\zeta).$$

Then the Doléans-Dade exponential $\mathcal{E}(U)$ is a martingale and the probability measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(U)_t,$$

is equivalent to $\mathbb{P}$. Moreover, under $\mathbb{Q}$, r is a CBI-Levy type process with the parameters $(a', b', \sigma', \gamma', \pi')$, where

$$a' = a - \sigma\,\eta - \gamma \int_0^\infty \zeta\,(e^{-\theta\zeta} - 1)\,\pi(d\zeta), \quad b' = ab/a', \quad \sigma' = \sigma, \quad \gamma' = \gamma$$

$$\pi'(d\zeta) = e^{-\theta\,\zeta}\,\pi(d\zeta).$$

Forward Pricing

In all the present section *the model parameters are assumed to be those defined by the risk-neutral dynamics*.

$$F(\tau, T) = \mathbb{E}^{\mathbb{Q}}\left[S(T) \,\middle|\, \mathcal{F}_\tau\right],$$

$$\begin{aligned}
F(\tau, T) &= B(T) + \sum_i \mathbb{E}^{\mathbb{Q}}\left[X_i(0) + \int_0^t a_i\left(b_i - X_i(s)\right) ds + \sigma_i \int_0^t \int_0^{X_i(s)} W_i(ds, du)\right. \\
&\quad \left. + \gamma_i \int_0^t \int_0^{X_i(s-)} \int_{\mathbb{R}+} \zeta\, \widetilde{N}(ds, du, d\zeta) \,\middle|\, \mathcal{F}_\tau\right]. \\
&= B(T) - B(\tau) + S(\tau) + \sum_i \frac{e^{-a_i(T-\tau)} - 1}{a_i}\left(X_i(\tau) - b_i\right)
\end{aligned}$$

Flow Forwards

If we denote by $[T_1, T_2]$ the delivery period, the value of the contract $F(\tau; T_1, T_2)$, $\tau < T_1$, is given by the following formulas :

$$
\begin{aligned}
F(\tau, T_1, T_2) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)du \middle| \mathcal{F}_\tau \right] = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathbb{E}^{\mathbb{Q}}\left[S(u) \middle| \mathcal{F}_\tau \right] du \\
&= S(\tau) - \alpha(\tau) - \sum_i \frac{e^{-a_i(T_2-\tau)} - e^{-a_i(T_1-\tau)} + a_i(T_2 - T_1)}{a_i^2(T_2 - T_1)} (X_i(\tau) - b_i) \\
&\quad + \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \alpha(u)du
\end{aligned}
$$

The Risk Premium

The risk premium is a relevant quantity in power markets description. We want then to provide an explicit representation formula for this quantity in the present modeling framework.
The risk premium can be defined as the difference between conditional expectations of the underlying price computed with respect to the risk-neutral measure $\mathbb{Q}$ and the historical measure $\mathbb{P}$:

$$R(\tau, T) = \mathbb{E}^{\mathbb{Q}}\left[S(T)|\mathcal{F}_\tau\right] - \mathbb{E}^{\mathbb{P}}\left[S(T)|\mathcal{F}_\tau\right].$$

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According to the results obtained in the previous section we can write :

$$R(\tau, T) = \sum_i \frac{e^{-A_i(T-\tau)} - 1}{A_i}\left(X_i(\tau) - B_i\right) - \sum_i \frac{e^{-a_i(T-\tau)} - 1}{a_i}\left(Y_i(\tau) - b_i\right),$$

where the parameters A_i, B_i are related to the parameters a_i and b_i by the relations describing the measure change :

$$A_i = a_i - \sigma_i\,\eta_i - \gamma_i \int_0^\infty \zeta\,(e^{-\theta_i\,\zeta} - 1)\,\mu_i(d\zeta), \quad B_i = \frac{a_i\,b_i}{A_i}$$

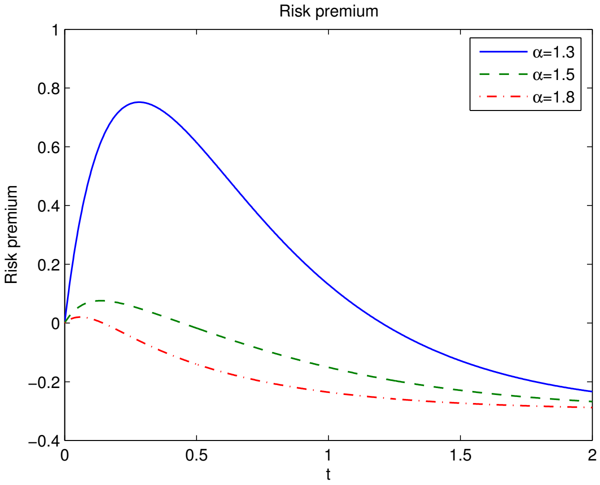

FIGURE – The Risk Premium Term Structure.

Calibration : Two-Factor Model

The first factor is continuous and corresponds to a standard CIR model and the second one is with jumps. Our objective is to make a thorough analysis of the jump behavior, in particular, for large jumps and spikes.

Let the first factor Y_1 be driven by a Gaussian random measure as

$$Y_1(t) = Y_1(0) + \int_0^t a_1 \left(b_1 - Y_1(s)\right) ds + \sigma_1 \int_0^t \int_0^{Y_1(s)} W_1(ds, du) \tag{4}$$

and the second factor Y_2 be driven by a pure jump Poisson random measure as

$$Y_2(t) = Y_2(0) + \int_0^t a_2 \left(b_2 - Y_2(s)\right) ds + \gamma_2 \int_0^t \int_0^{Y_2(s-)} \int_{\mathbb{R}+} \zeta \widetilde{N}_2(ds, du, d\zeta) \tag{5}$$

Spike factor

We are interested in the evolution of process Y_2 between two jump times, that is for any $t \in [\tau_k, \tau_{k+1})$,

$$Y_2(t) = Y_2(\tau_k) + \int_{\tau_k}^{t} a_2 \left(b_2 - Y_2(s)\right) ds + \gamma_2 \int_{\tau_k}^{t} \int_0^{Y_2(s-)} \int_{\mathbb{R}+} \zeta \widetilde{N}_2(ds, du, d\zeta).$$

With these notations, the time τ_{k+1} is the arrival time of the first jump after τ_k larger than z_0 for the measure N_2 or equivalently, larger than $\gamma_2 z_0$ for Y_2.

By the following result, we can separate small and large jumps and move the compensation inside the speed and mean coefficients.

The process Y_2 can be written, for all $t \in [\tau_k, \tau_{k+1})$, as

$$Y_2(t) = Y_2(\tau_k) + \int_{\tau_k}^{t} A_2 \left(B_2 - Y_2^{(z_0)}(s) \right) ds + \gamma_2 \int_{\tau_k}^{t} \int_0^{Y_2^{(z_0)}(s-)} \int_0^{z_0} \zeta \widetilde{N}_2(ds, du, d\zeta)$$
$$+\gamma_2 \int_{\tau_k}^{t} \int_0^{Y_2^{(z_0)}(s-)} \int_{z_0}^{\infty} \zeta N_2(ds, du, d\zeta),$$

where $z_0 > 0$ is a fixed constant,

$$A_2 := a_2 + \gamma_2 \int_{z_0}^{\infty} \zeta \mu(dz), \quad B_2 := \frac{a_2 b_2}{A_2}$$

and $Y_2^{(z_0)}$ is the truncated process, for all $t \in [\tau_k, \tau_{k+1})$, defined by

$$Y_2^{(z_0)}(t) = Y_2(\tau_k) + \int_{\tau_k}^{t} A_2 \left(B_2 - Y_2^{(z_0)}(s) \right) ds + \gamma_2 \int_{\tau_k}^{t} \int_0^{Y_2^{(z_0)}(s-)} \int_0^{z_0} \zeta \widetilde{N}_2(ds, du, d\zeta).$$

We stated that the truncated process $Y_2^{(z_0)}$ is linked to the intensity of the large jumps of Y_2. The following proposition explains in detail this link. Up to a constant, the process $Y_2^{(z_0)}$ is the stochastic intensity or *hazard rate* of the random time of the next big jump.
Let $\{\tau_k\}_{k\in\mathbb{N}}$ be defined by :

$$\tau_k^{(z_0)} = \inf\{t > \tau_{k-1}^{(z_0)} : \Delta Y_2(t) > \gamma_2 z_0\}, \quad \tau_0^{(z_0)} = 0.$$

Then we have

$$\mathbb{P}\left(\tau_{k+1} - \tau_k > t\right) = \mathbb{E}\left[exp\left\{-K_Y^{(z_0)} \int_{\tau_k}^{\tau_k+t} Y_2^{(z_0)}(s)ds\right\}\right],$$

where the renormalisation term $K_Y^{(z_0)} = \gamma_2 \int_{z_0}^{\infty} \zeta\mu_2(d\zeta)$ which is the proper truncated mass of the jumps distribution, and the frequency process $Y_2^{(z_0)}$ is given by the previous formula.

Finally, we deal with the asymptotic behaviour of $Y_2^{(z_0)}$ when the mean reverting speed a_2 diverges. Let us introduce the process $\widehat{Y}_2^{(z_0)}$ defined as

$$\widehat{Y}_2^{(z_0)}(t) = b_2 + e^{-A_2 t}\left[Y_2(0) - b_2\right] + \gamma_2 \int_0^t \int_0^{\widehat{Y}_2^{(z_0)}(s-)} \int_{z_0}^{\infty} e^{-A_2(t-s)} \zeta N_2(ds, du, d\zeta).$$

The next proposition shows that the two processes Y_2 and $Y_2^{(z_0)}$ have the same behaviour when a_2 goes to infinity. As a consequence, we can approximate the frequency of large jumps by the one of the Hawkes process as soon as a_2 is large. Then, if a_2 is large enough, the intensity of the jumps exhibits two behaviours. It is quite stable around b_2 but it jumps at all jumps times $\{\tau_k\}_{k\in\mathbb{N}}$ and exhibits an fast exponential decay to b_2 with speed $A_2 \geq a_2$. We have the following

Proposition

Consider Y_2 with $\mathbb{E}[Y_2(0)] < \infty$. As $a_2 \to +\infty$, we have that for each $t > 0$, $Y_2(t) - \widehat{Y}_2(t)$ goes to zero in probability.

First steps

Following the ideas presented in previous papers, the first step to perform is to de-seasonalise the data. The second step is to split the components Y_1 and Y_2 emerging from the data.

This issue is well analyzed in the papers by Beth, Kiesel and Nazarova (EE 2012) and their approach is directly applicable to our framework. Then, we first focus on the process Y_1, sometimes called the base signal.

We look for the ergodic distribution of Y_1 fitting the data. By recalling that the ergodic distribution of a CIR diffusion is of Gamma type, our model is in agreement with the previous literature and we obtain in a similar way the estimated parameters for Y_1.

Spike process

The estimation of the parameters of the spike process Y_2 is then our following main issue. Unfortunately, we cannot apply the techniques proposed since their model does not include the clustering effect that is crucial in our framework.

We remark that the process Y_2 is not directly observable since the data are given by the sum of three components, i.e. the seasonality function, the base signal and the spike process itself. Moreover, the great variance of the base signal covers the spike process far from the times of spikes.

That is the observation is reduced to the sequence $(\tau_k, \Delta S(\tau_k))_{k \in \mathbb{N}}$, $\Delta S(\tau_k) = S(\tau_k) - S(\tau_k^-)$, where τ_k is the time of the k^{th} spike and ΔS_{τ_k} is its jump size.

Due to the continuity of the seasonality function and the base signal, we have $\Delta S(\tau_k) = \Delta Y_2(\tau_k)$. As a consequence, we can assume that we observe the jump times and the jump sizes of Y_2.

Limit for large mean reverting speed

Our idea is then to estimate the intensity process $Y_2^{(z_0)}$ rather than Y_2 itself.
The mean-reverting speed is very high with respect to the one of the base signal as it has been pointed out in literature.

We may then consider that the limit distribution expressed as the approximate distribution of the jump frequency and we can then neglect small jumps.

In looking then at the sequence $(\tau_k, \Delta S(\tau_k))_{k\in\mathbb{N}}$, it can be considered as the realization of a marked Hawkes process N_2 with intensity $Y_2^{(z_0)}$. We remark that the parameters B_2, γ_2 and A_2 can then be estimated by the maximum likelihood estimator.

Likelihood function

Proposition

Given the observations $(\tau_k, \Delta S(\tau_k))_{k=1\dots N}$, we have the following Likelihood function

$$\log L\left(\tau_1, \Delta S(\tau_1), \dots \tau_N, \Delta S(\tau_N) | B_2, \gamma_2, A_2\right) := -B_2\tau_N + \sum_{i=1}^{N} \frac{\gamma_2 \Delta S(\tau_i)}{A_2}\left[e^{-A_2(\tau_N-\tau_i)} - 1\right] + \sum_{i=1}^{N} \log\left\{B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i-\tau_j)}\right\}$$

Moreover, the MLE estimators are :

$$
\begin{aligned}
\frac{\partial \log L}{\partial B_2} &= -\tau_N + \sum_{i=1}^{N} \left\{ B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)} \right\}^{-1} \\
\frac{\partial \log L}{\partial \gamma_2} &= \sum_{i=1}^{N} \frac{\Delta S(\tau_i)}{A_2} \left[e^{-A_2(\tau_n - \tau_i)} - 1 \right] + \sum_{i=1}^{N} \frac{\sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}}{B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}} \\
\frac{\partial \log L}{\partial A_2} &= \sum_{i=1}^{N} \frac{\gamma_2 \Delta S(\tau_i)}{A_2^2} \left\{ 1 - \left[A_2(\tau_n - \tau_i) + 1 \right] e^{-A_2(\tau_n - \tau_i)} \right\} \\
&\quad - \sum_{i=1}^{N} \frac{\gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j)(\tau_i - \tau_j) e^{-A_2(\tau_i - \tau_j)}}{B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}}
\end{aligned}
$$

Thank you for your attention !